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1992 J. Phys. A: Math. Gen. 25 5165

(http://iopscience.iop.org/0305-4470/25/19/024)

J. Phys. A: Math. Gen. 25 (1992) 5165–5176. Printed in the UK

Quantum diffusions, quantum dissipation and spin relaxation

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Received 25 February 1992

Abstract. We develop the tool of quantum diffusion (i.e. Hilbert-space-valued stochastic differential equations) for dissipative quantum systems. The aims are to find possible limitations to this approach and to investigate new pictures of open quantum systems. We are guided by the relaxation process for arbitrary spin and the associated natural rotational symmetry. We also impose the condition that the spin-coherent states remain coherent during the dissipative evolution. We present a new quantum diffusion equation that satisfies the above conditions and that is the unique quantum diffusion satisfying Percival's condition $(\mathrm{d}\psi)^2 = 0$.

1. Introduction

Dissipative quantum dynamics is usually described as a reduced dynamics of the system coupled to its environment [1, 2]. This approach goes back to Pauli [3] and describes the system of interest by a mixed density operator. Dissipative Schrödinger equations have also been considered, as pragmatic tools describing the system alone (see the following references: [4] for a review; [5–7] for concrete examples; [8] for applications to squids; [9] for applications to quantum tunnelling; [10] for applications to quantum optics). Such equations are deterministic; the system is then described at all times by one pure state.

Measurements are specific environments in which information is extracted from the system. This is as true for quantum systems as for classical ones. In the quantum case, the evolution of the system is usually described differently in such measurement-like situations: in addition to the density operator the system is also described by an explicit statistical mixture of pure states. In the ideal case, the latter is given by the projection postulate. There have recently been several attempts to describe the evolution of a quantum system during a measurement by means of stochastic equations in Hilbert spaces (quantum diffusions)—see [11–20].

This article is not about measurements, but we shall use the same kind of stochastic equations to describe dissipative quantum systems in general, and arbitrary spin relaxation in particular. After all, if this approach is meaningful for the case of measurement-like situations, still an open question, it should also apply to the general case of dissipative quantum dynamics.

0305-4470/92/195165+12$04.50

Let us be somewhat more specific. Let $\mathcal{S}$ represent the system under consideration and $\mathcal{E}$ its environment. If $\mathcal{S}$ and $\mathcal{E}$ interact, the state vector of $\mathcal{S}+\mathcal{E}$ will, according to the Schrödinger equation, describe a situation in which $\mathcal{S}$ and $\mathcal{E}$ are heavily entangled, like two spin-$\frac{1}{2}$ in the singlet state. Then all the usual problems about quantum non-locality, universal non-separability, the many worlds and its many interpretations, Bell's inequality and so on, follow. For instance, in the simple singlet-state case, the non-separability of the two spin-$\frac{1}{2}$ subsystems can be realized by registering the perfect correlation between measurements on each subsystem along parallel directions, or by measuring several correlations that violate the Bell inequality†. But this requires that the relative axis of one spin with respect to the other one is known: one must know what parallel means. If one spin undergoes some precession, the relative axis will change: perfect correlation will hold for some non-parallel axis. If one still knows the relation, then Bell's inequality can still be violated. If one loses track of the relation, then one can no longer design the experiment that would violate the Bell inequality. For the general case of a system $\mathcal{S}$ and its environment $\mathcal{E}$, the complexity is such that one almost immediately loses track of the precise relation between $\mathcal{S}$ and $\mathcal{E}$ [23, 24]. The discussion then turns around whether this should be interpreted as an entangled system but with hidden quantum correlations (perfect correlation still exists, but one does not know 'which pair of axes are parallel'), or whether it represents a separated system for which the quantum correlations with the environment have been broken (no longer any correlation). From a practical point of view it amounts to the same thing. But let us make two remarks at this point. First a question: is it acceptable that the interpretation of our fundamental physical theory is based on the fact that it is practically impossible to find the error implicit in interpreting an extremely complicated entangled state as a mixture of separated (i.e. product) state? Second, if one tries to describe seriously the disentanglement of $\mathcal{S}+\mathcal{E}$, one is led to investigate some theories close to, but different from, quantum mechanics. In this way, limitations to this type of theory can be found [18, 25] and different predictions from quantum mechanics can be evaluated with the aim of designing some crucial experimental tests.

This has motivated Ghirardi *et al* [26] and Bell [20] (see also Pearle [11, 12], Penrose [27], Károlyházy [28], Shimony [29], Diósi [13–15], and Gisin [17, 18]) to promote the idea that quantum dynamics has to be completed by stochastic terms in the dynamics. The Ghirardi–Rimini–Weber model is the most developed of such 'completed pictures'. We do not see any good reason why the picture should be changed only for the case of measurement-like situations (i.e. conservative diffusions); we like to investigate the consequences of a more radical change, somewhat in the spirit of Percival's new 'correspondence principle' [30]. One of Ghirardi–Rimini–Weber merits is that their proposal is valid at all times, not only for specific situations. However their proposal breaks most of the symmetries of quantum mechanics: q is strongly differentiated from p, spins are even stranger than before, etc. So the idea here is to apply the existing tools for the study of arbitrary evolutions of density operators and their corresponding Hilbert space diffusions.

Actually, as we shall see, there are too many possible stochastic equations corresponding to a given dissipative evolution equation for density matrices [25], that is stochastic equations such that the average of the pure states over the 'noise' equals the mixed density matrix. We shall show that the one proposed in [16, 18] lacks the

† Note that such a violation occurs for any non-product state [21, 22].

natural symmetry that one might expect for spin relaxation. We shall also be guided by the assumption that the 'correct' equation preserves coherent states, that is an initially spin-coherent state should remain coherent.

The non-Hamiltonian terms that can appear on the right-hand side of an evolution equation for density operators ρ_t are of the form (assuming complete positivity [31]) $B\rho_t B^+ - \frac{1}{2}\{B^+B, \rho_t\}$ where B is a linear operator. If B is self-adjoint, then two very different kinds of quantum diffusions can reproduce such terms on average. First, one with a fluctuating Hamiltonian $\mathrm{d}\psi_t = -\mathrm{i}B\psi_t \circ \mathrm{d}W_t$ (the $\circ$ denotes the use of a Stratonovich equation) with the standard Wiener process $(\mathrm{d}W_t)^2 = \mathrm{d}t$; then the average over W_t of the projector $\psi_t\psi_t^+$ follows such an equation [17]. The other quantum diffusion localizes the state vectors on the eigenspaces of B (see e.g. [18]) i.e. the quantum average $\langle B\rangle_{\psi_t}$ changes with time, contrary to the first quantum diffusion, but in such proportions that the average over the noise of $\psi_t\psi_t^+$ follows the same equation. If B is not self-adjoint, then no stochastic Hamiltonian can reproduce the density-matrix evolution, but there are more general stochastic equations for pure states whose average reproduces the density matrix. This article is about these latter terms and the corresponding dissipative quantum diffusions.

In the next section we shall consider the spin-$\frac{1}{2}$ case and impose rotational symmetry. In section 3 spin-coherent states are presented for completeness, and section 4 treats the case of deterministic friction. This prepares for the main section, section 5, where arbitrary spin relaxation is considered under the assumption that the coherent states are preserved by the (stochastic) evolution. The article ends, as usual, with some conclusions and perspectives, discussed in section 6.

2. Example of the relaxation of spin $\frac{1}{2}$ with rotational symmetry

The dissipative evolution equation for the density matrix ρ_t representing a spin $\frac{1}{2}$ in a magnetic field at zero temperature reads†:

$$\dot{\rho}_t = 2\sigma_+\rho_t\sigma_- - \{\sigma_-\sigma_+, \rho_t\} \tag{2.1}$$

where $\sigma_\pm = (\sigma_x + \mathrm{i}\sigma_y)/2$, $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. The only stationary solution of (2.1) is spin-up and all states tend asymptotically to this stationary state. The corresponding quantum diffusion proposed in [16] and [18] is given by the following Itô stochastic equation:

$$\mathrm{d}\psi_t = \sqrt{2}(\sigma_+ - \langle\sigma_+\rangle_{\psi_t})\psi_t \mathrm{d}\xi_t - (\sigma_-\sigma_+ - 2\langle\sigma_-\rangle_{\psi_t}\sigma_+ + \langle\sigma_-\rangle_{\psi_t}\langle\sigma_+\rangle_{\psi_t})\psi_t \mathrm{d}t \tag{2.2}$$

where $\langle\sigma_\pm\rangle_{\psi_t} = \langle\psi_t|\sigma_\pm|\psi_t\rangle / \langle\psi_t|\psi_t\rangle$ and the Wiener process $\mathrm{d}\xi_t$ satisfies $\mathrm{d}\xi_t^2 = \mathrm{d}t$. It is straightforward to verify that (2.2) preserves the norm of ψ_t: $d|\psi_t| = 0$ and from here on we assume that ψ_t is normalized. The only stationary solution of (2.2) is, again, spin-up. Furthermore the corresponding one-dimensional projector $P_t = \psi_t^+\psi_t$ satisfies

$$\mathrm{d}P_t = \sqrt{2}[(\sigma_+ - \langle\sigma_+\rangle_{\psi_t})P_t + P_t(\sigma_- - \langle\sigma_-\rangle_{\psi_t})]\mathrm{d}\xi_t + [2\sigma_+P_t\sigma_- - \{\sigma_-\sigma_+, P_t\}]\mathrm{d}t\,.$$

† At finite temperature an additional term describes fluctuations via a fluctuating magnetic field.

Note that the last term of this equation (the drift) equals the right-hand side of (2.1). Consequently the average over the Wiener process of the one-dimensional projector (pure state) $\psi_t^+\psi_t$ equals ρ_t.

The equation (2.1) is manifestly invariant under rotation around the z-direction and this symmetry is physically obvious. But (2.2) lacks this symmetry, since σ_+ depends on the choice of x- and y-axis. A way to recover the rotational invariance is to add to (2.2) a similar term, with a second independent Wiener process, with axes rotated by 90°:

$$\sigma_+ \longrightarrow \mathrm{e}^{\mathrm{i}\pi\sigma_z/4}\sigma_+\mathrm{e}^{-\mathrm{i}\pi\sigma_z/4} = \mathrm{i}\sigma_+ \,.$$

Hence the rotational invariant equation corresponding to (2.1) is

$$\begin{aligned} \mathrm{d}\psi_t = {} & \left(\sigma_+ - \langle\sigma_+\rangle_{\psi_t}\right)\psi_t\left(\mathrm{d}\xi_t + \mathrm{i}\,\mathrm{d}W_t\right) \\ & - \left(\sigma_-\sigma_+ - 2\langle\sigma_-\rangle_{\psi_t}\sigma_+ + \langle\sigma_-\rangle_{\psi_t}\langle\sigma_+\rangle_{\psi_t}\right)\psi_t\,\mathrm{d}t \end{aligned} \tag{2.3}$$

where $\mathrm{d}\xi_t^2 = \mathrm{d}W_t^2 = \mathrm{d}t$, $\mathrm{d}\xi_t\mathrm{d}W_t = 0$. Its Stratonovich form is particularly simple:

$$\mathrm{d}\psi_t = (\sigma_+ - \langle\sigma_+\rangle_{\psi_t})\psi_t \circ (\mathrm{d}\xi_t + \mathrm{i}\,\mathrm{d}W_t + 2\langle\sigma_-\rangle_{\psi_t}\mathrm{d}t) - (\sigma_-\sigma_+ - \langle\sigma_-\sigma_+\rangle_{\psi_t})\psi_t\mathrm{d}t \,. \tag{2.4}$$

If θ_t and φ_t represent the angles defining the direction $\langle\sigma\rangle_{\psi_t}$ then, with $\eta_t = \cos\theta_t$, we have:

$$\langle\sigma\rangle_{\psi_t} = \left(\sqrt{1-\eta_t^2}\cos\varphi_t, \sqrt{1-\eta_t^2}\sin\varphi_t, \eta_t\right)$$

$$\mathrm{d}\eta_t = 2(1-\eta_t)\mathrm{d}t + (1-\eta_t)\sqrt{1-\eta_t^2}\,\mathrm{d}\xi_t^* \qquad \mathrm{d}\varphi_t = -\sqrt{\frac{1-\eta_t}{1+\eta_t}}\,\mathrm{d}W_t^*$$

where

$$\mathrm{d}\xi_t^* = \cos\varphi_t\;\mathrm{d}\xi_t - \sin\varphi_t\;\mathrm{d}W_t \qquad \mathrm{d}W_t^* = \sin\varphi_t\;\mathrm{d}\xi_t + \cos\varphi_t\;\mathrm{d}W_t$$

i.e. $(\mathrm{d}\xi_t^*)^2 = (\mathrm{d}W_t^*)^2 = \mathrm{d}t$, $\mathrm{d}\xi_t^*\mathrm{d}W_t^* = 0$. This equation corresponds to $f = \sqrt{(1-\eta_t)/(1+\eta_t)}$ of our classification [25].

It is interesting to note that (2.3), contrary to (2.2), satisfies a condition introduced by Percival [30] from considerations about uniqueness of the quantum diffusion equations, namely $\mathrm{d}\psi_t \otimes \mathrm{d}\psi_t = 0$. We thus recover Percival's condition starting from physical considerations about symmetries.

3. Spin-coherent states—a reminder

Here we briefly recall the main aspects of the notion of spin-coherent states. We refer to [32–34] for more information on the extensive work connected with these states. They are the analogues for spin systems of the well known coherent states for the harmonic oscillator [35] and they are often used to develop a phase-space approach to spin systems using functions on the sphere [36, 37].

Let $S = (S_x, S_y, S_z)$ be the spin-s operators acting on $\mathbf{C}^{2s+1}$ and satisfying the usual commutations relations $[S_j, S_k] = \mathrm{i}\varepsilon_{jkl} S_l$. We choose and fix a value of $s \in \{\frac{1}{2}, 1, \frac{3}{2}, \ldots\}$ and we denote by $|m\rangle$ the S_z eigenstates labelled by their corresponding eigenvalue ($S_z|m\rangle = m|m\rangle$, $m = -s, -s+1, \ldots, s-1, s$). A spin-$s$ coherent state $|n\rangle$ is then determined by a unit vector

$$n = \left(\sqrt{1-\eta^2}\cos\varphi, \sqrt{1-\eta^2}\sin\varphi, \eta\right) \qquad \eta = \cos\theta$$

of the two-dimensional sphere S^2. More precisely, if we take the eigenstate $|s\rangle$ as a fundamental vector the spin-s coherent state $|n\rangle$ is defined by [32, 34]

$$|n\rangle = R(n)|s\rangle \tag{3.1}$$

where $R(n) = \mathrm{e}^{-\mathrm{i}\theta k\cdot S}$ is the unitary representation of a rotation through the angle θ about the axis $k = (-\sin\varphi, \cos\varphi, 0)$ orthogonal both to n and $e_z = (0,0,1)$. This rotation is the simplest one which transports e_z onto n and, clearly, this definition of k is valid for any n excluding that corresponding to the south pole $n = (0, 0, -1)$. Thus, in accordance with the general reasoning, the spin-coherent states are indexed by points of S^2 which may therefore be considered as the phase space of the 'classical' spin.

Using the stereographic projection of the sphere onto the C-plane through the south pole, $n \longrightarrow \zeta = \mathrm{e}^{\mathrm{i}\varphi}\tan(\theta/2)$, one can express $|n\rangle$ in terms of the S_z eigenstates and of the angular variables (θ, φ). The explicit form that one obtains is [32, 34]

$$|n\rangle = 2^{-s}\sum_{m=-s}^{s}\sqrt{\binom{2s}{s-m}}\mathrm{e}^{\mathrm{i}(s-m)\varphi}(1-\eta)^{(s-m)/2}(1+\eta)^{(s+m)/2}|m\rangle\,. \tag{3.2}$$

Hence, in our representation, $|n\rangle$ is singular at the south pole whereas the state $|e_z\rangle$, with e_z pointing upwards, is naturally identified with the spin-up state $|s\rangle$. Furthermore, the system of spin-coherent states is over-complete and from (3.2) one deduces [36]

$$\langle n'|n\rangle = \left[\frac{(1 + e_z\cdot n' + e_z\cdot n + n\cdot n' + \mathrm{i}e_z\cdot n'\wedge n)^2}{4(1+e_z\cdot n')(1+e_z\cdot n)}\right]^s .$$

This clearly shows that the states $|n\rangle$ are normalized but are not mutually orthogonal. For completeness we add the two following formulas which are derived from (3.2):

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}|n\rangle = \mathrm{i}(s - S_z)|n\rangle \qquad \frac{\mathrm{d}}{\mathrm{d}\eta}|n\rangle = \frac{S_z - \mathrm{i}s\eta}{1-\eta^2}|n\rangle\,.$$

We recall also that a spin-coherent state can be characterized by any of the following equivalent properties:

$$n\cdot S|n\rangle = s|n\rangle \tag{3.3a}$$

$$\langle S\rangle_n = sn \tag{3.3b}$$

$$\sum_{k=x,y,z}\langle S_k\rangle_n^2 = s^2 \tag{3.3c}$$

$$\sum_{k=x,y,z}(\Delta S_k)^2 = s \tag{3.3d}$$

where $\langle \boldsymbol{S}\rangle_{\boldsymbol{n}} \equiv \langle \boldsymbol{n}|\boldsymbol{S}|\boldsymbol{n}\rangle$ and $(\Delta S_k)^2 \equiv \langle S_k^2\rangle_{\boldsymbol{n}} - \langle S_k\rangle_{\boldsymbol{n}}^2$. In other words we have: (i) the spin-coherent state $|\boldsymbol{n}\rangle$ is an eigenvector for the operator $\boldsymbol{n}\cdot\boldsymbol{S}$ with maximal eigenvalue; (ii) the expectation value of the $\boldsymbol{S}$ operator on coherent state $|\boldsymbol{n}\rangle$ is the vector of length s pointing in the $\boldsymbol{n}$ direction; (iii) the spin-coherent states minimize the sum of the Heisenberg uncertainties. The expressions (3.3*a*–*d*) can be deduced from the definition (3.1) and the transformation law $R(\boldsymbol{n})S_z R(\boldsymbol{n})^{-1} = \boldsymbol{n}\cdot\boldsymbol{S}$ [32–34].

Finally, to conclude this reminder, we point out that it is possible to obtain explicit formulae for expectation values of very general spin-observables on the set of coherent states $|\boldsymbol{n}\rangle$. This can be done, for instance, by applying the formalism developed in [36]. In particular, for latter use, we compute the $|\boldsymbol{n}\rangle$ expectation value of the anticommutator $\{\boldsymbol{a}\cdot\boldsymbol{S}, \boldsymbol{b}\cdot\boldsymbol{S}\}$ where $\boldsymbol{a}$ and $\boldsymbol{b}$ are arbitrary three-dimensional vectors. One gets:

$$\langle\{\boldsymbol{a}\cdot\boldsymbol{S},\boldsymbol{b}\cdot\boldsymbol{S}\}\rangle_{\boldsymbol{n}} = s(2s-1)(\boldsymbol{a}\cdot\boldsymbol{n})(\boldsymbol{b}\cdot\boldsymbol{n}) + s\boldsymbol{a}\cdot\boldsymbol{b}\,. \tag{3.4}$$

This expression is easily obtained using the differential form of the spin-operator introduced in [36]. More precisely, in [36] it is proved that one has for any spin-observable A

$$\langle \boldsymbol{S}A\rangle_{\boldsymbol{n}} = \boldsymbol{I}(\boldsymbol{n})\langle A\rangle_{\boldsymbol{n}} \tag{3.5}$$

where $\boldsymbol{I}(\boldsymbol{n})$ is the following first-order differential operator

$$\boldsymbol{I}(\boldsymbol{n}) = s\boldsymbol{n} - \tfrac{\mathrm{i}}{2}\boldsymbol{n}\wedge\boldsymbol{\nabla}_{\boldsymbol{n}} - \tfrac{1}{2}\boldsymbol{n}\wedge(\boldsymbol{n}\wedge\boldsymbol{\nabla}_{\boldsymbol{n}})\,. \tag{3.6}$$

Here $\boldsymbol{\nabla}_{\boldsymbol{n}}$ denotes the gradient with respect to the variables (n_x, n_y, n_z) and (3.5) means that the $|\boldsymbol{n}\rangle$ expectation value of the product $\boldsymbol{S}A$ is obtained directly by applying the operator $\boldsymbol{I}(\boldsymbol{n})$ on the $|\boldsymbol{n}\rangle$ expectation value of A which is a well defined function on the sphere S^2. The adaptation of this result to our context gives

$$\begin{aligned}\langle\{\boldsymbol{a}\cdot\boldsymbol{S},\boldsymbol{b}\cdot\boldsymbol{S}\}\rangle_{\boldsymbol{n}} &= \langle(\boldsymbol{a}\cdot\boldsymbol{S})(\boldsymbol{b}\cdot\boldsymbol{S})\rangle_{\boldsymbol{n}} + \langle(\boldsymbol{b}\cdot\boldsymbol{S})(\boldsymbol{a}\cdot\boldsymbol{S})\rangle_{\boldsymbol{n}}\\ &= s\left[\boldsymbol{a}\cdot\left(\sum_{k=x,y,z} b_k\boldsymbol{I}(\boldsymbol{n})n_k\right) + \boldsymbol{b}\cdot\left(\sum_{k=x,y,z} a_k\boldsymbol{I}(\boldsymbol{n})n_k\right)\right].\end{aligned} \tag{3.7}$$

From the definition (3.6), a straightforward calculation yields

$$\sum_{k=x,y,z} a_k\boldsymbol{I}(\boldsymbol{n})n_k = \tfrac{1}{2}[\,(2s-1)(\boldsymbol{a}\cdot\boldsymbol{n})\boldsymbol{n} - \mathrm{i}(\boldsymbol{n}\wedge\boldsymbol{a}) + \boldsymbol{a}] \tag{3.8}$$

and an equivalent formula with the vector $\boldsymbol{b}$ instead of $\boldsymbol{a}$. Then one immediately finds the right-hand side of (3.4) by inserting (3.8) in (3.7). Choosing now $\boldsymbol{a} = \boldsymbol{e}_k$ and $\boldsymbol{b} = \boldsymbol{e}_l$ in (3.4), $\boldsymbol{e}_k$ and $\boldsymbol{e}_l$ being the unit vectors along the k and l axis, one gets the useful formula for the anticommutator between S_k and S_l

$$\langle\{S_k, S_l\}\rangle_{\boldsymbol{n}} = s(2s-1)n_k n_l + s\delta_{kl} \tag{3.9}$$

and, in particular, when $l = k$ one finds

$$\langle S_k^2\rangle_{\boldsymbol{n}} = \frac{s}{2}\left[1 + (2s-1)n_k^2\right]. \tag{3.10}$$

4. The deterministic spin-relaxation example

Before studying the case of stochastic relaxation we present in this section a result for a deterministic dissipative equation. This result will be useful for the next section. The deterministic dissipative equation is motivated by simplicity [7], by its natural implementation in Hilbert space formalisms [38] and by the models that have been developed based on it [9, 10, 39].

The equation considered in this section reads

$$\dot{\psi}_t = \left(\langle S_z \rangle_{\psi_t} - S_z\right) \psi_t \tag{4.1}$$

and describes the relaxation of a spin s. This can be seen by noticing that from (4.1) one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle S_z \rangle_{\psi_t} = -2\left(\langle S_z^2 \rangle_{\psi_t} - \langle S_z \rangle_{\psi_t}^2\right) \leqslant 0 \,.$$

We prove the following result:

Theorem. Assume that the evolution of the state ψ_t is governed by (4.1). If, at time $t = 0$, $\psi_0 \equiv |\boldsymbol{n}_0\rangle$ is a spin-s coherent state, then:

(i) the state remains coherent during the entire evolution: $\psi_t = |\boldsymbol{n}(t)\rangle$ for all t;

(ii) the evolution equation for the unit vector

$$\boldsymbol{n}(t) = \left(\sqrt{1-\eta_t^2}\cos\varphi_t, \sqrt{1-\eta_t^2}\sin\varphi_t, \eta_t\right) \in \mathcal{S}^2 \tag{4.2}$$

which determines the spin-coherent state at time t is given by

$$\dot{\boldsymbol{n}}(t) = \boldsymbol{n}(t) \wedge [\boldsymbol{n}(t) \wedge \boldsymbol{e}_z] \tag{4.3}$$

or in terms of angular variables

$$\dot{\varphi}_t = 0 \qquad \dot{\eta}_t = \eta_t^2 - 1 \,.$$

Remarks.

(i) The equation (4.3) has been introduced by Landau and Lifschitz for a phenomenological description of magnetic moment relaxation [40, 41].

(ii) Note that if the right-hand side of (4.1) is multiplied by a complex-valued time-dependent function, then the first conclusion of the theorem remains valid. Indeed, this is obvious for the real part of the function and, for the imaginary part, it follows from the well known fact that rotations must preserve spin-coherent states [32].

(iii) If in (4.1) we substitute S_z by the harmonic oscillator Hamiltonian a^+a, then the resulting equation preserves the harmonic coherent states [7].

Proof. Let us define

$$F(t) \equiv \sum_{k=x,y,z} \langle S_k \rangle_{\psi_t}^2 \,.$$

First, one remarks that from (3.3*c*) one has

$$\psi_t = |\boldsymbol{n}(t)\rangle \Longleftrightarrow F(t) = s^2 . \tag{4.4}$$

Since, by hypothesis, $\psi_0 \equiv |\boldsymbol{n}_0\rangle$ is initially a spin-coherent state, (4.4) implies that $F(0) = s^2$. Hence in order to prove the first assertion of the theorem it suffices to show that $F(t)$ remains constant during the evolution. With this end in view, we remark that if ψ_t evolves according to (4.1), then for every τ such that $F(\tau) = s^2$ one has $\dot{F}(\tau) = 0$. Indeed, from (4.1) a straightforward calculation yields

$$\dot{F}(t) = 2 \sum_{k=x,y,z} \langle S_k\rangle_{\psi_t} \left(2\langle S_k\rangle_{\psi_t}\langle S_z\rangle_{\psi_t} - \langle\{S_k, S_z\}\rangle_{\psi_t}\right) . \tag{4.5}$$

If $F(\tau) = s^2$, it follows by (4.4) that ψ_τ is given by a spin-coherent state $|\boldsymbol{n}(\tau)\rangle$. Then one can calculate (4.5) at time τ by computing explicitly the $\boldsymbol{n}(\tau)$ expectation values $\langle\cdot\rangle_{\boldsymbol{n}(\tau)}$ from the expressions (3.9) and (3.3*b*). One gets:

$$\begin{aligned}\dot{F}(\tau) = 2 &\sum_{k=x,y,z} s n_k(\tau)[2s^2 n_k(\tau) n_z(\tau) - s(2s-1) n_k(\tau) n_z(\tau) - s\delta_{kz}] \\ = \; 2s^2 &\sum_{k=x,y,z} n_k(\tau)[n_k(\tau) n_z(\tau) - \delta_{kz}] = 0 .\end{aligned}$$

From this result and the fact that $F(0) = s^2$, it obviously follows that $F(t) = s^2$ for all t. Together with (4.4) this implies $\psi_t = |\boldsymbol{n}(t)\rangle$ for all t and proves the first part of the theorem. The second assertion follows from a straightforward calculation which uses (3.3*b*) and (4.1). One has:

$$\dot{\boldsymbol{n}}(t) = \frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{S}\rangle_{\boldsymbol{n}(t)} = \frac{1}{s}\left[2\langle \boldsymbol{S}\rangle_{\boldsymbol{n}(t)}\langle S_z\rangle_{\boldsymbol{n}(t)} - \langle\{\boldsymbol{S}, S_z\}\rangle_{\boldsymbol{n}(t)}\right] . \tag{4.6}$$

Inserting the explicit formulae (3.9) and (3.3*b*) into (4.6) one obtains the following set of equations for the three components of the vector $\boldsymbol{n}(t)$:

$$\dot{n}_x(t) = n_x(t) n_z(t), \; \dot{n}_y(t) = n_y(t) n_z(t) \qquad \dot{n}_z(t) = n_z(t)^2 - 1 .$$

These equations can be expressed in a more compact form as

$$\dot{\boldsymbol{n}}(t) = \boldsymbol{n}(t) \wedge [\boldsymbol{n}(t) \wedge \boldsymbol{e}_z] .$$

5. Quantum diffusion for arbitrary spin relaxations

All spin-$\frac{1}{2}$ states are coherent but, obviously, this is not the case for arbitrary spin s. In this section we introduce quantum diffusion equations for arbitrary spin-s relaxation that preserve coherent states. This means that the solution of the stochastic equation for an initially coherent state is itself coherent (with probability 1) at all times.

First let us notice that neither the generalization to arbitrary spin of (2.2) nor (2.4) preserve coherent states. This is easily seen on the Stratonovich form of the equation since the Stratonovich product follows the same rules as ordinary analysis. The first term of (2.4) has the same form as the deterministic spin relaxation (4.1) studied in the previous section and thus does preserve coherent states (see the second remark after the theorem of section 4). However, the second term has a different form, involving not only S_k operators, but also S_k^2 operators. A compensation for this last term is then necessary; this can be done and the result is contained in the next theorem.

Theorem. Assume that the evolution of the state ψ_t is governed by the following Stratonovich stochastic differential equation:

$$\begin{aligned} \mathrm{d}\psi_t = {} & \left(S_+ - \langle S_+\rangle_{\psi_t}\right) \psi_t \circ \left(\mathrm{d}\xi_t + \mathrm{i}\,\mathrm{d}W_t + 2\langle S_-\rangle_{\psi_t}\mathrm{d}t\right) \\ & - \left(S_-S_+ - \langle S_-S_+\rangle_{\psi_t}\right) \psi_t \mathrm{d}t \\ & + \left(S_z - \langle S_z\rangle_{\psi_t}\right) \psi_t \circ \left(\sqrt{2}\mathrm{d}\chi_t + 2\langle S_z\rangle_{\psi_t}\mathrm{d}t\right) - \left(S_z^2 - \langle S_z^2\rangle_{\psi_t}\right) \psi_t \mathrm{d}t \end{aligned} \tag{5.1}$$

where ξ_t, W_t, and χ_t are three independent Wiener processes with $\mathrm{d}\xi_t^2 = \mathrm{d}W_t^2 = \mathrm{d}\chi_t^2 = \mathrm{d}t$. If, at time $t = 0$, $\psi_0 \equiv |\boldsymbol{n}_0\rangle$ is a spin-s coherent state, then:

(i) the evolution equation for the density matrix ρ_t is

$$\dot{\rho}_t = 2S_+\rho_t S_- - \{S_-S_+, \rho_t\} - \tfrac{1}{2}[S_z[S_z, \rho_t]] \tag{5.2}$$

(ii) the state remains coherent during the entire evolution: $\psi_t = |\boldsymbol{n}(t)\rangle$ for all t where $\boldsymbol{n}(t)$ is defined by (4.2);

(iii) one has:

$$\mathrm{d}\eta_t = \sqrt{2}(1-\eta_t^2)\mathrm{d}\chi_t + (1-\eta_t)\sqrt{1-\eta_t^2}\,\mathrm{d}\xi_t^* + 2\left[1 - \eta_t + (s - \tfrac{1}{2})(1-\eta_t^2)\right]\mathrm{d}t \tag{5.3}$$

$$\mathrm{d}\varphi_t = -\sqrt{\frac{1-\eta_t}{1+\eta_t}}\,\mathrm{d}W_t^* \tag{5.4}$$

where

$$\mathrm{d}\xi_t^* = \cos\varphi_t \mathrm{d}\xi_t - \sin\varphi_t \mathrm{d}W_t \tag{5.5a}$$

$$\mathrm{d}W_t^* = \sin\varphi_t \mathrm{d}\xi_t + \cos\varphi_t \mathrm{d}W_t \tag{5.5b}$$

i.e. $(\mathrm{d}\xi_t^*)^2 = (\mathrm{d}W_t^*)^2 = \mathrm{d}t$, $\mathrm{d}\xi_t^*\mathrm{d}W_t^* = 0$.

Remarks.

(i) Note that the above χ_t noise term could also be complexified as $\mathrm{d}\chi_t + \mathrm{i}\,\mathrm{d}\tilde{\chi}_t$. The same conclusion about preservation of coherent states holds, since the imaginary part would simply impose a rotation around the z-axis with a stochastic angular velocity.

(ii) For the harmonic oscillator, if one replaces in (5.1) the operators S_+ by the creation operator a^+, S_- by the annihilation operator a and $S_z = \frac{1}{2}[S_-, S_+]$ by $[a, a^+]$, that is by a multiple of the identity, then the equation corresponding to (5.1) also preserves the (harmonic oscillator) coherent states [18].

Proof.

(i) Equation (5.2) is a direct consequence of [18] and of the independence of the three Wiener processes ξ_t, W_t and χ_t.

(ii) To prove the second assertion of the theorem it suffices to remark that since $S_-S_+ + S_z^2 = s(s+1) - S_z$, all terms in (5.1) have the same form as the deterministic spin relaxation (4.1). Thus, the same proof as the theorem of section 4 holds here since the Stratonovich product is used in (5.1).

(iii) To obtain (5.3), let us first compute the drift term b of $\mathrm{d}\eta_t$ (that is the term multiplying $\mathrm{d}t$). Since (5.3) is written in the Itô form, its drift term equals the evolution of its average:

$$b = \frac{\mathrm{d}}{\mathrm{d}t}\langle\langle \eta_t \rangle\rangle_{\xi_t, W_t, \chi_t}\,.$$

But the latter is given by the evolution (5.2) of the density matrix. Hence, using the cyclic property of the trace we have

$$b = \frac{1}{s}[2\langle S_- S_z S_+\rangle_{\boldsymbol{n}(t)} - \langle\{S_z, S_- S_+\}\rangle_{\boldsymbol{n}(t)}]$$

and the expression found in (5.3) for b follows from a straightforward calculation which uses (3.3*b*) and (3.10). For the stochastic terms of (5.3) we first compute the coefficient of $\mathrm{d}\chi_t$ (note that for the stochastic terms one need not distinguish between Itô and Stratonovich equations, since the transformation from the one to the other affects only the drift term). This computation is identical to the one applied to (4.5) in section 4. One has

$$\frac{2\sqrt{2}}{s}[\langle S_z^2\rangle_{\boldsymbol{n}(t)} - \langle S_z\rangle^2_{\boldsymbol{n}(t)}]\,\mathrm{d}\chi_t$$

and one gets the expression which appears in (5.3) using (3.3b) and (3.10) again. For the coefficient of $\mathrm{d}\xi_t^*$, the computation is similar. One gets

$$\frac{1}{s}[\langle S_z S_+\rangle_{\boldsymbol{n}(t)} - \langle S_z\rangle_{\boldsymbol{n}(t)}\langle S_+\rangle_{\boldsymbol{n}(t)}]\;[\mathrm{d}\xi_t + \mathrm{i}\,\mathrm{d}W_t]$$
$$+ \frac{1}{s}[\langle S_- S_z\rangle_{\boldsymbol{n}(t)} - \langle S_z\rangle_{\boldsymbol{n}(t)}\langle S_-\rangle_{\boldsymbol{n}(t)}]\;[\mathrm{d}\xi_t - \mathrm{i}\,\mathrm{d}W_t]\,.$$

As before, using (3.3*b*), (3.9) and the definition (5.5*a*) one finds the expression in (5.3). Finally, the computation of $\mathrm{d}\varphi_t$ to prove (5.4) is lengthy but without difficulty, keeping in mind that $\tan\varphi_t = \langle S_y\rangle_{\boldsymbol{n}(t)}/\langle S_x\rangle_{\boldsymbol{n}(t)}$, and using (5.5*b*).

6. Conclusions

We have seen that because of symmetry reasons and in order that the evolution preserves coherent states we must introduce complex-valued Wiener processes. The equation associated to a Lindblad term

$$2B\rho_t B^+ - \{B^+B, \rho_t\} \tag{6.1}$$

for any linear operator B is thus

$$\mathrm{d}\psi_t = \left(B - \langle B\rangle_{\psi_t}\right)\psi_t \circ \left(\mathrm{d}\xi_t + \mathrm{i}\,\mathrm{d}W_t + 2\langle B^+\rangle_{\psi_t}\mathrm{d}t\right) - \left(B^+B - \langle B^+B\rangle_{\psi_t}\right)\psi_t\mathrm{d}t\,. \tag{6.2}$$

This equation satisfies $(\mathrm{d}\psi_t)^2 = 0$. Percival, following Diósi, has proven that the correspondence between (6.1) and a Hilbert space diffusion is unique if one imposes the condition $(\mathrm{d}\psi_t)^2 = 0$. Equation (6.2) (possibly with a sum over several operators B_i with independent Wiener processes) is thus the explicit form of the diffusion characterized by Diósi and Percival. Although this equation was derived in the context of spin relaxation, the other standard example of dissipative quantum dynamics, that is the damped harmonic oscillator, also fits nicely in this frame. Indeed, defining $B = a$ (the annihilation operator), and adding the usual Hamiltonian term $-\mathrm{i}a^+a\psi_t\mathrm{d}t$ one obtains a description of the damped quantum oscillator that respects the p, q symmetry and that preserves the coherent states.

Equation (6.2) looks highly nonlinear. Let us show that this is not so. The nonlinear terms used to shift the operator B and B^+B to zero mean value affect only the norm of ψ_t. They are chosen such that the norm $|\psi_t|$ remains constant. But one can work with unnormalized state vectors ψ_t (for finite times, so that the norm does remain strictly positive), then the nonlinearity in (6.2) is a shift of the C-valued noise.

Note that for such a non-norm-preserving equation

$$\mathrm{d}\psi_t = B\psi_t(\mathrm{d}\xi_t + \mathrm{i}\,\mathrm{d}W_t + 2\langle B^+\rangle_{\psi_t}\mathrm{d}t) - B^+B\psi_t\mathrm{d}t$$

the Stratonovich and the Itô equations have exactly the same relatively simple form†.

The nonlinear term in this last equation can still be removed with a Girsanov transformation [43]. One then has a linear equation which is the analogue for our (6.2) of Pearle's 'raw ensemble' [12].

Note also that in the case of self-adjoint operators B, our investigation suggests that the two quantum diffusions that we mentioned in our introduction should be mixed in equal proportion.

It is our hope that (6.2) will turn out to be simpler and physically more instructive than (6.1) in some specific practical cases. We look for fresh intuition for the description of open quantum systems. Our approach could be specially illustrative and useful in numerical computations of nonlinear quantum dynamics of open systems [44]. We also expect to apply it to situations in which measurements on open quantum systems are crucial, like in the experimental tests of the Zeno paradox [45], the recent experiments on quantum jumps [46] in quantum optics and to the many forthcoming experiments that become feasible thanks to the recent and ongoing breakthrough in technology.

Acknowledgments

It is a pleasure to thank Ian Percival and Abner Shimony for many interesting discussions and their careful reading of the manuscript. MBC is supported by the Swiss National Science Foundation.

† After completion of this work Philip Pearle informed us that he came up with this same equation in a different context [42].

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